Causal Inference Methods and Case Studies

STAT24630

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Lecture 8

Topic: post-stratification, pairwise randomized experiment

- Post-stratification
- pairwise randomized experiment
  - Fisher’s exact p-value
  - Neyman’s repeated sampling approach
  - Regression analysis
  - How to find strata / pairs?

Post-stratification

- In a completely randomized experiment, each assignment vector has the sample probability ($P(\boldsymbol{W} = \boldsymbol{w})$) if $\sum_{i=1}^{N} w_i = N_t$
- If we focus on a subgroup S, conditional on $N_{t,S} = \sum_{i \in S} W_i$, the assignment vector for the individuals in the subgroup also has the same probability ($P(\boldsymbol{W}_S = \boldsymbol{w}_S)$) if $\sum_{i \in S} w_i = N_{t,S}$
- So conditional on $N_{t,S}$, we can treat the treatment assignment as from a completely randomized experiment also for the subgroup

- Post-stratification (Miratrix. et al. 1971. J. Royal Stat. Soc. B.)
  - Blocking after the experiment is conducted
  - Analyze the experiment as from a stratified randomized experiment by conditioning on $N_{t,S}$ for each strata S
  - By post-stratification, we can stratify individuals into relatively homogenous subpopulations
  - Post-stratification is nearly as efficient as pre-randomization blocking except with a large number of small strata

Meinert et. al. (1970)'s example

- A completely randomized experiment.
- Treatment is tolbutamide ($Z = 1$) and control is a placebo ($Z = 0$)
- Causal effect: difference in the survival probability

	Age < 55			Age ≥ 55	
	Surviving	Dead		Surviving	Dead
$Z = 1$	98	8	$Z = 1$	76	22
$Z = 0$	115	5	$Z = 0$	69	16

Total	Surviving	Dead
$Z = 1$	174	30
$Z = 0$	184	21

Peng's book Section 5.4.1

- Subgroup and sample average estimates with post-stratification

	stratum 1	stratum 2	post-stratification	crude
est	−0.034	−0.036	−0.035	−0.045
se	0.031	0.060	0.032	0.033

Pairwise randomized experiment

- Procedure:
  1. Create $J = N/2$ pairs of similar units
  2. Randomize treatment assignment within each pair

- Assignment probability
  A special case of stratified randomized experiment where $N(j) = 2$ and $N_t(j) = 1$

$$P(\boldsymbol{W} = \boldsymbol{w}|\boldsymbol{X}) = \begin{cases} \prod_{j=1}^{J} \binom{N(j)}{N_t(j)}^{-1} = 2^{-N/2} & \text{if } \sum_{i:B_i=j}^{N} w_i = 1 \text{ for } j = 1, \cdots, J \\ 0 & \text{otherwise} \end{cases}$$

The Children’s television workshop experiment
[Ball, Bogatz, Rubin and Beaton, 1973.]

- The Educational Testing Service (ETS) wanted to evaluate *The Electric Company,* an American educational children's television series aimed at improving reading skills for young children

- Two sites, Yongstown, Ohio and Fresno, California where the show was not broadcast on local television, were selected to evaluate the effect of watching the show at school

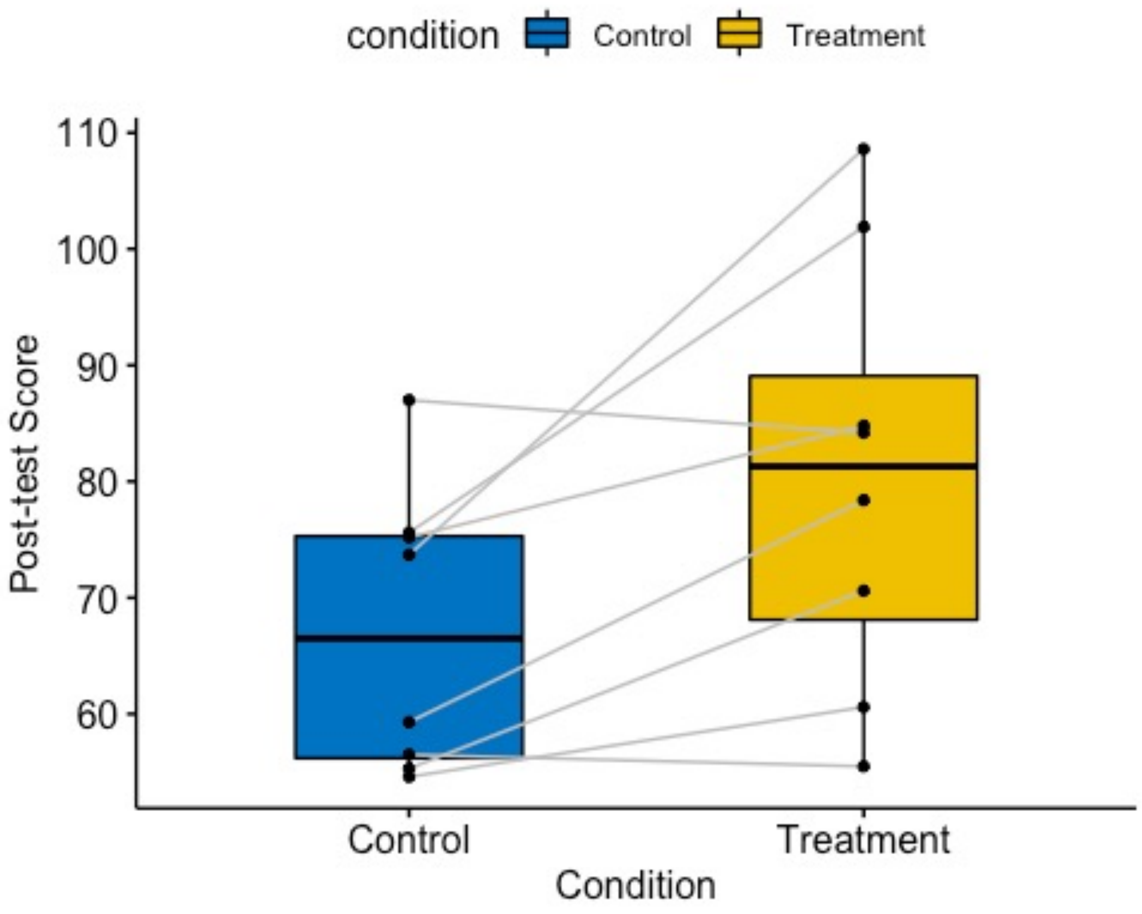
- Within each school, a pair of two classes are selected
  - One class randomly assigned to watch the show
  - Another class continue with regular reading curriculum

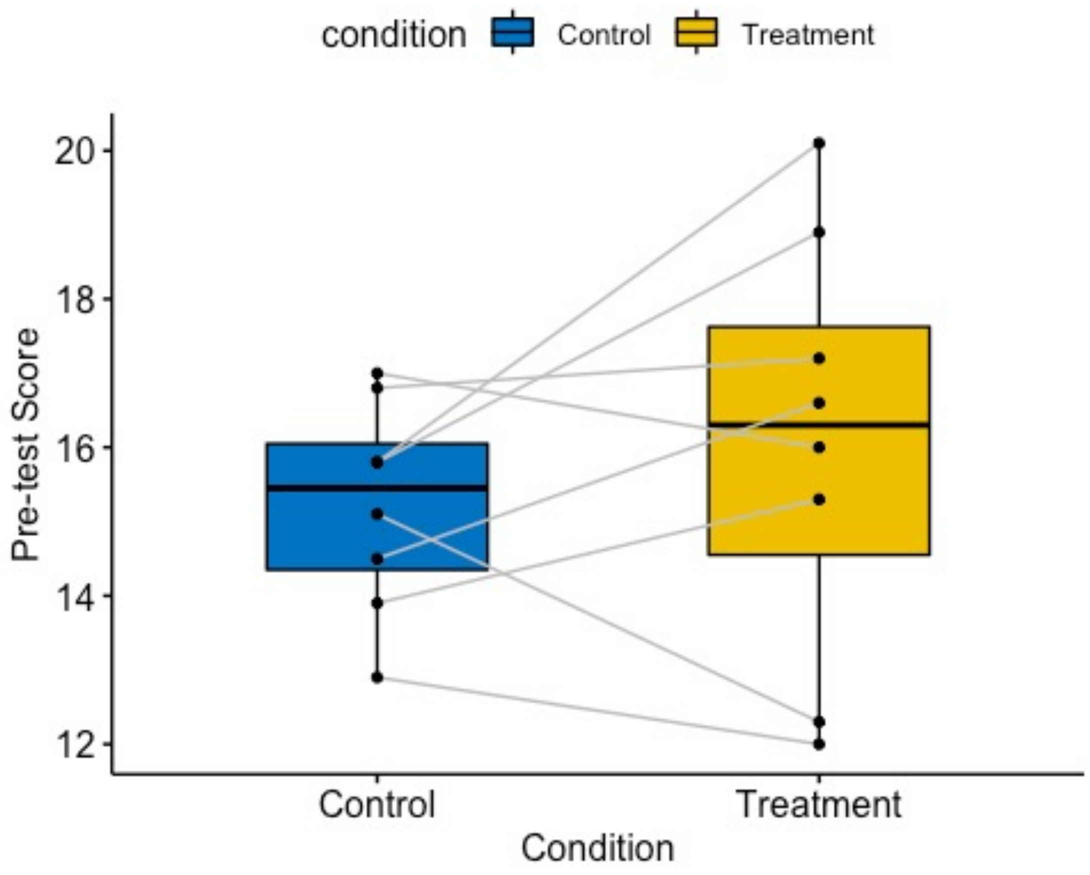
Data from Youngstown

Pair G_i	Treatment W_i	Pre-Test Score X_i	Post-Test Score Y_i^{obs}
1	0	12.9	54.6
1	1	12.0	60.6
2	0	15.1	56.5
2	1	12.3	55.5
3	0	16.8	75.2
3	1	17.2	84.8
4	0	15.8	75.6
4	1	18.9	101.9
5	0	13.9	55.3
5	1	15.3	70.6
6	0	14.5	59.3
6	1	16.6	78.4
7	0	17.0	87.0
7	1	16.0	84.2
8	0	15.8	73.7
8	1	20.1	108.6

- Two first-grade classes from each of eight schools participate in the experiment
- ETS performed reading ability tests to the kids both before the program started and after it finished.

Data from Youngstown





Some notations

Pair	Unit A					Unit B				
	$Y_{i,A}(0)$	$Y_{i,A}(1)$	$W_{i,A}$	$Y_{i,A}^{\text{obs}}$	$X_{i,A}$	$Y_{i,B}(0)$	$Y_{i,B}(1)$	$W_{i,B}$	$Y_{i,B}^{\text{obs}}$	$X_{i,B}$
1	54.6	?	0	54.6	12.9	?	60.6	1	60.6	12.0
2	56.5	?	0	56.5	15.1	?	55.5	1	55.5	13.9
3	75.2	?	0	75.2	16.8	?	84.8	1	84.8	17.2
4	76.6	?	0	75.6	15.8	?	101.9	1	101.9	18.9
5	55.3	?	0	55.3	13.9	?	70.6	1	70.6	15.3
6	59.3	?	0	59.3	14.5	?	78.4	1	78.4	16.6
7	87.0	?	0	87.0	17.0	?	84.2	1	84.2	16.0
8	73.7	?	0	73.7	15.8	?	108.6	1	108.6	20.1

- Average treatment effect within pair *j*

$$\tau^{\text{pair}}(j) = \frac{1}{2}\sum_{i:G_i=j}\big(Y_i(1) - Y_i(0)\big) = \frac{1}{2}\Big(\big(Y_{j,A}(1) - Y_{j,A}(0)\big) + \big(Y_{j,B}(1) - Y_{j,B}(0)\big)\Big).$$

- Observed outcomes for both treatment and control groups

$$Y_{j,c}^{\text{obs}} = \begin{cases} Y_{j,A}^{\text{obs}} & \text{if } W_{i,A} = 0, \\ Y_{j,B}^{\text{obs}} & \text{if } W_{i,A} = 1, \end{cases} \quad \text{and} \quad Y_{j,t}^{\text{obs}} = \begin{cases} Y_{j,B}^{\text{obs}} & \text{if } W_{i,A} = 0, \\ Y_{j,A}^{\text{obs}} & \text{if } W_{i,A} = 1. \end{cases}$$

Fisher's exact p-value

- We still focus on the **Sharp null:** $H_0: Y_i(0) \equiv Y_i(1)$ for all $i = 1, \cdots, N$
- Choice of test statistics:
  - Average group mean differences across pairs

$$T^{\text{dif}} = \left| \frac{1}{J} \sum_{j=1}^{J} \left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} \right) \right| = \left| \bar{Y}_t^{\text{obs}} - \bar{Y}_c^{\text{obs}} \right|$$

    As each pair has exactly one treatment and one control
    - We don't need to consider different weights
    - No worry of Simpson's paradox

  - Rank statistics
    - Use population ranks: $T = |\overline{\text{rank}}(Y_t^{\text{obs}}) - \overline{\text{rank}}(Y_c^{\text{obs}})|$
    - Use within-pair ranks

$$T^{\text{rank,pair}} = \left| \frac{2}{N} \sum_{j=1}^{N/2} \left(\mathbf{1}_{Y_{j,1}^{\text{obs}} > Y_{j,0}^{\text{obs}}} - \mathbf{1}_{Y_{j,1}^{\text{obs}} < Y_{j,0}^{\text{obs}}} \right) \right|$$

Application to the television workshop data

- Fisher's exact p-values
  - Mean differences: $T = 13.4$, $\text{pvalue} = 0.031$
  - Rank mean differences: $T = 3.75$, $\text{pvalue} = 0.031$
  - Within-pair rank differences: $T = 0.5$, $\text{pvalue} = 0.29$

- Rank v.s. within-pair rank
  - Both can reduce the sensitivity to outliers
  - Using within-pair ranks can have more power when there is substantial variation in the level of the outcomes between pairs
  - Otherwise, using within-pair ranks loses power as it treats small within-pair differences (which may be due to random noises) equally with large within-pair differences

  - Using within-pair ranks is more appropriate for large, heterogenous population

Neyman's repeated sampling approach

- Target: PATE or SATE $\tau = \sum_j \frac{N(j)}{N}\tau(j)$ where $\tau(j)$ is the PATE or SATE for strata j

- Point estimate:

$$\hat{\tau}^{\text{pair}}(j) = Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} \qquad \hat{\tau}^{\text{dif}} = \frac{1}{N/2}\sum_{j=1}^{N/2}\hat{\tau}^{\text{pair}}(j) = \overline{Y}_{\text{t}}^{\text{obs}} - \overline{Y}_{\text{c}}^{\text{obs}}$$

- We can not estimate the within-pairs variances as there are only two units per pair
- Use the following empirical estimate of the uncertainty (paired t-test)

$$\hat{\mathbb{V}}^{\text{pair}}\left(\hat{\tau}^{\text{dif}}\right) = \frac{4}{N\cdot(N-2)}\cdot\sum_{j=1}^{N/2}\left(\hat{\tau}^{\text{pair}}(j) - \hat{\tau}^{\text{dif}}\right)^2$$

- Above estimate is conservative

$$\mathbb{E}\left[\hat{\mathbb{V}}^{\text{pair}}\left(\hat{\tau}^{\text{dif}}\right)\right] = \mathbb{V}_W(\hat{\tau}^{\text{dif}}) + \frac{4}{N\cdot(N-2)}\cdot\sum_{j=1}^{N/2}\left(\tau^{\text{pair}}(j) - \tau\right)^2$$

Application to the television workshop data

- Est. = 13.4, sd. = 4.6, 95% CI: $[4.3, 22.5]$
- As we have 8 pairs, Gaussian approximation is inaccurate and it's better to compare with a t-distribution with df = 7
- 95% CI comparing with t-distribution: $[2.5, 24.3]$
- If we treat the data as from completely randomized experiment, then sd. = 7.8

Pair	Outcome for Control Unit	Outcome for Treated Unit	Difference
1	54.6	60.6	6.0
2	56.5	55.5	−1.0
3	75.2	84.8	9.6
4	75.6	101.9	26.3
5	55.3	70.6	15.3
6	59.3	78.4	19.1
7	87.0	84.2	−2.8
8	73.7	108.6	34.9
Mean	67.2	80.6	13.4
(S.D.)	(12.2)	(18.6)	(13.1)

Linear regression

- We can not run separate linear regressions within each pair, as there are only 2 units per pair

- We assume that $Y_i(w) = \alpha_j + \tau_i w + \boldsymbol{\beta}^T \boldsymbol{X}_i + \varepsilon_i^*$ where $\mathbb{E}(\tau_i - \tau \mid \boldsymbol{X}_i) = \boldsymbol{\gamma}^T (\boldsymbol{X}_i - \overline{\boldsymbol{X}})$
- Then we have
$$\mathbb{E}\left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \boldsymbol{W} = \boldsymbol{w}, \boldsymbol{X} = \boldsymbol{x}\right) = \tau + \boldsymbol{\gamma}^T \left(\overline{\boldsymbol{X}}_j - \overline{\boldsymbol{X}}\right) + \left(\boldsymbol{\beta} + \frac{\boldsymbol{\gamma}}{2}\right)^T \left(\boldsymbol{X}_{j,t} - \boldsymbol{X}_{j,c}\right)$$
where $\boldsymbol{X}_{j,t}$ and $\boldsymbol{X}_{j,c}$ are the covariates for the treated and control unit of the jth pair, and $\overline{\boldsymbol{X}}_j$ is the average between the two
- τ is still the PATE
- We still implicitly condition on the pair indicators variables
- If $\boldsymbol{\gamma} = \boldsymbol{0}$, then $\mathbb{E}\left(Y_{j,t}^{\text{obs}} - Y_{j,c}^{\text{obs}} | \boldsymbol{W} = \boldsymbol{w}, \boldsymbol{X} = \boldsymbol{x}\right) = \tau + \boldsymbol{\beta}^T \left(\boldsymbol{X}_{j,t} - \boldsymbol{X}_{j,c}\right)$ we only need to include the covariates differences in the linear regression model

- We can assume homoscedastic errors in the linear regression even if $\mathbb{V}(Y_i(0)) \neq \mathbb{V}(Y_i(1))$

How to perform stratification / pairing

- Univariate blocking: discrete or discretized variable
- Multivariate blocking: Mahalanobis distance

$$D(\mathbf{X}_i, \mathbf{X}_j) = \sqrt{(\mathbf{X}_i - \mathbf{X}_j)^\top \widehat{\mathbb{V}(\mathbf{X})}^{-1} (\mathbf{X}_i - \mathbf{X}_j)}$$

Greedy algorithms

- Matching: pair two units with the shortest distance, set them aside, and repeat
- Blocking: randomly choose one unit and choose N_j units with the shortest distances, set them aside, and repeat

But the resulting matches may not be optimal

Optimal matching

- D: $N \times N$ matrix of pairwise distance or a cost matrix
- Select N elements of D such that there is only one element in each row and one element in each column and the sum of pairwise distances is minimized

- Linear Sum Assignment Problem (LSAP)
  - Binary $N \times N$ matching matrix: M with $M_{ij} \in \{0,1\}$
  - Optimization problem

$$\min_M \sum_{i=1}^{N} \sum_{j=1}^{N} M_{ij} D_{ij} \quad \text{subject to} \sum_{i=1}^{N} M_{ij} = 1, \sum_{j=1}^{N} M_{ij} = 1$$

    where we set $D_{ii} = \infty$ for all i
  - can apply the Hungarian algorithm

Example: evaluation of health insurance policy

[Public policy for the poor? A randomised assessment of the Mexican universal health insurance programme. *The lancet, 2009*.]

- Seguro Popular, a programme aimed to deliver health insurance, regular and preventive medical care, medicines, and health facilities to 50 million uninsured Mexicans
- Units: health clusters = predefined health facility catchment areas
- 4 pre-treatment cluster-average covariates: age, education, household size, household assets
- 100 clusters, 50 pairs